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Interpretation and extension of the Green's ansatz for paraparticles

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Abstract

The anomalous bilinear commutation relations satisfied by the components of the Green's ansatz for paraparticles are shown to derive from the comultiplication of the paraboson or parafermion algebra. The same provides a generalization of the ansatz, wherein paraparticles of order $p = \sum_{\alpha=1}^{r} p_\alpha$ are constructed from r paraparticles of order p_α, $\alpha = 1, 2, \ldots, r$.

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In the last few years, there has been increasing interest in generalized statistics, different from Bose and Fermi statistics. The main reason is their possible relevance to the theory of the fractional quantum Hall effect [1], to that of anyon superconductivity [2], and to the description of black hole statistics [3]. There have been various proposals, among which we may quote parastatistics [4, 5], anyon statistics [6], quon statistics [7], Haldane fractional statistics [8], and also some recent attempts to provide a unified view of some of them (see e.g. [9]).

In the present Letter, we shall be concerned with Green's parabose and parafermi statistics [4], which arose from a remark of Wigner back in 1950 [10], and were among the first consistent examples of generalized statistics. Green's parastatistics is based upon trilinear commutation relations for particle creation and annihilation operators. It is characterized by a discrete parameter $p \in \mathbb{N}$, called the order of paraquantization and interpolating between Bose and Fermi statistics. For the paraboson (resp. parafermion) case, only those representations of the symmetric group S_N with at most p rows (resp. columns) do occur, so that at most p parabosons (resp. parafermions) can be in an antisymmetric (resp. symmetric) state.

Systems made of a single type of parabosons can be alternatively described [11, 12] in the framework of the Calogero-Vasiliev algebra [13] Fock space representation. This algebra also plays a crucial role [14] in understanding the algebraic properties of the two-particle Calogero problem [15]. While showing an interesting connection between parastatistics and the fast-growing field of integrable models, the equivalence established in Refs. [11] and [12] also provides a characterization of parabosons in terms of bilinear commutation relations, hence a more convenient approach to the Fock space construction.

To deal with the latter problem for systems made of more than one type of parabosons or for systems of parafermions, one still has to resort to the Green's ansatz [4], expressing parabosons (resp. parafermions) of order p as combinations of p anticommuting bosons (resp. commuting fermions). A natural question that arises in connection with such a construction is why the boson or fermion operators partially commute and partially anticommute. To avoid this problem, it has been proposed to consider ordinary, i.e., commuting (resp. anticommuting), bosons (resp. fermions) and to multiply them either by Clifford matrices [16],

or by Majorana fermions [12]. In another connection, it has been argued that the Green's ansatz construction is a very natural method from the Lie superalgebra viewpoint [17].

Here we shall adopt another viewpoint, which will at the same time provide an explanation for the strange behaviour of the Green's ansatz components and generalize the construction. It is based on the addition of paraboson (resp. parafermion) operators or, in mathematical terms, on the comultiplication of the paraboson (resp. parafermion) algebra, which is part of the Hopf algebraic structure of the latter [18].

Let us recall that the paraboson or parafermion algebra is generated by n pairs of creation and annihilation operators $a_k^\dagger$, a_k, $k = 1, 2, \ldots, n$, satisfying the trilinear commutation relations [4]

$$\begin{aligned}
\left[a_k, \left[a_l^\dagger, a_m\right]_\pm\right]_- &= 2\delta_{kl} a_m, \\
\left[a_k, \left[a_l^\dagger, a_m^\dagger\right]_\pm\right]_- &= 2\delta_{kl} a_m^\dagger \pm 2\delta_{km} a_l^\dagger, \\
\left[a_k, \left[a_l, a_m\right]_\pm\right]_- &= 0,
\end{aligned} \tag{1}$$

where (and in what follows) the upper and lower signs refer to parabosons and parafermions, respectively, and as usual $[x, y]_\pm \equiv xy \pm yx$.

A question that arises is how to build paraboson (resp. parafermion) operators $a_k^\dagger$, a_k, i.e., operators satisfying Eq. (1), out of two commuting (resp. anticommuting) sets of paraboson (resp. parafermion) operators $\left(a_{1k}^\dagger, a_{1k}\right)$ and $\left(a_{2k}^\dagger, a_{2k}\right)$, both satisfying Eq. (1). It is important to stress here that by demanding that $[a_{1k}, a_{2l}]_\mp = \left[a_{1k}, a_{2l}^\dagger\right]_\mp = 0$, we endow the paraboson (resp. parafermion) operators with an even (resp. odd) character. This seems quite natural having in mind the special case of bosons (resp. fermions).

If we had a Lie algebra or superalgebra, the answer to the question would be simple, since we would have $a_k^\dagger = a_{1k}^\dagger + a_{2k}^\dagger$, $a_k = a_{1k} + a_{2k}$.[1] In mathematical terms, this would mean that the coproducts $\Delta\left(a_k^\dagger\right) = a_k^\dagger \otimes I + I \otimes a_k^\dagger$, $\Delta(a_k) = a_k \otimes I + I \otimes a_k$ would fulfil the same relations as $a_k^\dagger$, a_k. Here, the symbol $\otimes$ denotes standard tensor product in the Lie algebraic case, but supertensor product in the Lie superalgebraic one, i.e., $(x \otimes y)(z \otimes t) = (-1)^{|y||z|}(xz \otimes yt)$,

[1] This is valid for instance for boson operators (see e.g. [19]).

where $|y|$ and $|z|$ are the degrees of y and z, respectively (see e.g. examples 1.5.7 and 10.1.3 in Ref. [20]).

For the algebra (1), the problem is more complicated, but was solved by Daskaloyannis *et al.* [18] in the paraboson case. In the parafermion one, these authors assumed that the operators of the two sets $\left(a_{1k}^\dagger, a_{1k}\right)$ and $\left(a_{2k}^\dagger, a_{2k}\right)$ commute with one another, so that we shall not follow their solution. Actually, it is straightforward to see that provided we assume that the two sets of parafermion operators anticommute, the same comultiplication (and more generally the same Hopf structure) is valid for parabosons and parafermions.

To define the comultiplication, it is necessary to first extend the paraboson or parafermion algebra with the operators

$$K = \exp(\mathrm{i}\pi\mathcal{N}), \qquad K^\dagger = \exp(-\mathrm{i}\pi\mathcal{N}), \tag{2}$$

where

$$\mathcal{N} = \frac{1}{2}\sum_{k=1}^{n}\left[a_k^\dagger, a_k\right]_\pm . \tag{3}$$

As a consequence of Eq. (1), they fulfil the relations

$$\begin{aligned} KK^\dagger &= K^\dagger K = I, \\ \left[K, a_k^\dagger\right]_+ &= \left[K, a_k\right]_+ = \left[K^\dagger, a_k^\dagger\right]_+ = \left[K^\dagger, a_k\right]_+ = 0. \end{aligned} \tag{4}$$

It is then a simple matter to check that the operators

$$\Delta(a_k) = a_k \otimes I + K \otimes a_k, \qquad \Delta(a_k^\dagger) = a_k^\dagger \otimes I + K^\dagger \otimes a_k^\dagger, \tag{5}$$

$$\Delta(K) = K \otimes K, \qquad \Delta\left(K^\dagger\right) = K^\dagger \otimes K^\dagger, \tag{6}$$

satisfy both the trilinear relations (1) and the additional relations (4) if

$$[a_k \otimes I, I \otimes a_l]_\mp = \left[a_k \otimes I, I \otimes a_l^\dagger\right]_\mp = 0, \tag{7}$$

which is consistent with the even (resp. odd) character of paraboson (resp. parafermion) operators.

Equation (1) only determines the parabosonic or parafermionic nature of the creation and annihilation operators. To fix the order of paraquantization p, one has to impose the additional condition

$$a_k a_l^\dagger |0\rangle = \delta_{kl}\, p\, |0\rangle, \tag{8}$$

where $|0\rangle$ is the parabosonic or parfermionic vacuum state, i.e.,

$$a_k |0\rangle = 0. \tag{9}$$

Let us assume that we start from two sets of paraboson or parfermion operators $\left(a_{1k}^\dagger, a_{1k}\right)$, $\left(a_{2k}^\dagger, a_{2k}\right)$ with fixed orders of paraquantization p_1 and p_2, respectively, or in other words that

$$a_{\alpha k} a_{\alpha l}^\dagger |0\rangle_\alpha = \delta_{kl}\, p_\alpha\, |0\rangle_\alpha, \qquad a_{\alpha k}|0\rangle_\alpha = 0, \qquad \alpha = 1, 2. \tag{10}$$

What can we say from their combination? By using Eq. (5), we obtain that $|0\rangle_1 |0\rangle_2 = |0\rangle \otimes |0\rangle$ is the vacuum state of the combined operators,

$$\Delta(a_k)(|0\rangle \otimes |0\rangle) = 0, \tag{11}$$

and moreover that

$$\Delta(a_k)\Delta(a_l^\dagger)(|0\rangle \otimes |0\rangle) = \delta_{kl}(p_1 + p_2)(|0\rangle \otimes |0\rangle). \tag{12}$$

This shows that the combined operators are parabosons or parafermions of order $p_1 + p_2$.

The results obtained so far for the addition of two types of paraboson or parafermion operators can be easily extended to that of r types of such operators. This implies iterating the comultiplication defined in Eqs. (5) and (6). Such a process is made possible by an important property of the comultiplication valid for coalgebras (and more generally for Hopf algebras), called coassociativity, according to which the order wherein the addition of three types of operators is performed does not matter:

$$(\mathrm{id} \otimes \Delta)\Delta(X) = (\Delta \otimes \mathrm{id})\Delta(X). \tag{13}$$

It is straightforward to check that Eqs. (5) and (6) indeed satisfy condition (13).

Hence, we may recursively define $\Delta^{(r-1)} \equiv \left(\Delta \otimes I^{(r-2)}\right) \Delta^{(r-2)}$, $\Delta^{(1)} \equiv \Delta$, where $I^{(r-2)} \equiv I \otimes I \otimes \cdots \otimes I$ $(r-2$ times), and construct the operators

$$\Delta^{(r-1)}(a_k) = \sum_{\alpha=1}^{r} K^{(\alpha-1)} \otimes a_k \otimes I^{(r-\alpha)}, \qquad \Delta^{(r-1)}\left(a_k^\dagger\right) = \sum_{\alpha=1}^{r} \left(K^\dagger\right)^{(\alpha-1)} \otimes a_k^\dagger \otimes I^{(r-\alpha)}, \quad (14)$$

and

$$\Delta^{(r-1)}(K) = K^{(r)}, \qquad \Delta^{(r-1)}\left(K^\dagger\right) = \left(K^\dagger\right)^{(r)}, \qquad (15)$$

satisfying Eqs. (1) and (4), with $K^{(\alpha)} \equiv K \otimes K \otimes \cdots \otimes K$, $\left(K^\dagger\right)^{(\alpha)} \equiv \left(K^\dagger\right) \otimes \left(K^\dagger\right) \otimes \cdots \otimes \left(K^\dagger\right)$ (α times).

Equations (11) and (12) can also be extended to

$$\Delta^{(r-1)}(a_k)\, |0\rangle^{(r)} = 0, \qquad (16)$$

$$\Delta^{(r-1)}(a_k) \Delta^{(r-1)}\left(a_l^\dagger\right) |0\rangle^{(r)} = \delta_{kl} \left(\sum_{\alpha=1}^{r} p_\alpha\right) |0\rangle^{(r)}, \qquad (17)$$

where $|0\rangle^{(r)} \equiv |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$ (r times), and the operators $a_k^\dagger$, a_k belonging to the αth subspace of the tensor product (or the αth set) are assumed to be of order p_α.

Let us now consider the r components of the iterated coproducts of a_k and $a_k^\dagger$, defined in Eq. (14),

$$a_k^{(\alpha)} = K^{(\alpha-1)} \otimes a_k \otimes I^{(r-\alpha)}, \qquad a_k^{(\alpha)\dagger} = \left(K^\dagger\right)^{(\alpha-1)} \otimes a_k^\dagger \otimes I^{(r-\alpha)}, \qquad \alpha = 1, 2, \ldots, r. \quad (18)$$

In the special case where the operators $a_k^\dagger$, a_k are boson or fermion operators, i.e., $p_\alpha = 1$, $\alpha = 1, 2, \ldots, r$, it results from Eq. (18) and from the properties of the operators a_k, $a_k^\dagger$, K, $K^\dagger$ that

$$\begin{aligned} \left[a_k^{(\alpha)}, a_l^{(\alpha)\dagger}\right]_\mp &= \delta_{kl}\, I^{(r)}, \qquad \left[a_k^{(\alpha)}, a_l^{(\alpha)}\right]_\mp = 0, & (19) \\ \left[a_k^{(\alpha)}, a_l^{(\beta)\dagger}\right]_\pm &= \left[a_k^{(\alpha)}, a_l^{(\beta)}\right]_\pm = 0 \qquad (\alpha \neq \beta), & (20) \end{aligned}$$

and

$$a_k^{(\alpha)} |0\rangle^{(r)} = 0. \qquad (21)$$

Hence, the components $a_k^{(\alpha)}$, $a_k^{(\alpha)\dagger}$ of the iterated coproducts satisfy the same anomalous bilinear commutation relations as the components of the Green's ansatz: in the paraboson

(resp. parafermion) case, $a_k^{(\alpha)\dagger}$, $a_k^{(\alpha)}$ are boson (resp. fermion) creation and annihilation operators, but different sets of operators anticommute (resp. commute) among themselves. This shows that the iterated coproducts $\Delta^{(p-1)}(a_k)$, $\Delta^{(p-1)}\left(a_k^\dagger\right)$, where we now assume $r = p$, are but a realization of the Green's ansatz for paraparticles.

Furthermore, Eq. (18) allows us to derive a more general result. In the case where the p_α's are arbitrary, we indeed find that Eq. (19) has to be replaced by

$$\begin{aligned} \left[a_k^{(\alpha)}, \left[a_l^{(\alpha\dagger}, a_m^{(\alpha)}\right]_\pm\right] &= 2\delta_{kl} a_m^{(\alpha)}, \\ \left[a_k^{(\alpha)}, \left[a_l^{(\alpha\dagger}, a_m^{(\alpha\dagger)}\right]_\pm\right] &= 2\delta_{kl} a_m^{(\alpha)\dagger} \pm 2\delta_{km} a_l^{(\alpha)\dagger}, \\ \left[a_k^{(\alpha)}, \left[a_l^{(\alpha}, a_m^{(\alpha)}\right]_\pm\right] &= 0, \end{aligned} \tag{22}$$

while Eq. (20) remains valid, and Eq. (21) is supplemented by

$$a_k^{(\alpha)} a_l^{(\alpha)\dagger} |0\rangle^{(r)} = \delta_{kl}\, p_\alpha |0\rangle^{(r)}. \tag{23}$$

Paraboson (resp. parafermion) operators of order $p = \sum_{\alpha=1}^{r} p_\alpha$ can therefore be constructed from r anticommuting (resp. commuting) sets of paraboson (resp. parafermion) operators of order p_1, p_2, $\ldots$, p_r, respectively.

In conclusion, we did prove that the Green's ansatz for paraparticles has its origin in the Hopf algebraic structure of the paraboson or parafermion algebra. In such a context, the fact that this algebra is not a Lie algebra, but is defined in terms of trilinear relations, plays a crucial role. In addition, we showed that the ansatz is but a special case of a more general one, where bosons (resp. fermions) are replaced by parabosons (resp. parafermions) of order p_α, $\alpha = 1, 2, \ldots, r$, with $p = \sum_{\alpha=1}^{r} p_\alpha$.

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